Haar bases for $L^2(\mathbb{Q}_2^2)$ generated by one wavelet function [1]

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Abstract

The concept of p-adic quincunx Haar MRA was introduced and studied in [6]. In contrast to the real setting, infinitely many different wavelet bases are generated by a p-adic MRA. We give an explicit description for all wavelet functions corresponding to the quincunx Haar MRA. Each one generates an orthogonal basis, one of them was presented in [6]. A connection between quincunx Haar bases and two-dimensional separable Haar MRA is also found.

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1 Introduction

In 2002, S. V. Kozyrev [13] found a p-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the real Haar basis. V. M. Shelkovich and one of the authors developed an MRA approach to p-adic wavelets [17]. The scheme was realized to construct the p-adic Haar MRA. In contrast to the real setting, it appears

that there exist infinitely many different orthonormal wavelet bases with the minimal number of generating wavelet functions in the same Haar MRA. All these bases were described explicitly in [17] for $p = 2$ and in [11] for arbitrary p, one of these bases (for each p) coincides with Kozyrev's wavelet basis. A wide class of orthogonal scaling functions generating an MRA was described in [10] by A. Khrennikov, V. Shelkovuch and one of the authors. However it has been shown in [1] by S. Evdokimov and both authors that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA, except for those described in [10]. Thus all univariate orthogonal MRA-based wavelet bases with a minimal possible number of wavelet functions are described. It is not known if other p-adic orthogonal wavelet bases exist. Although, for the construction p-adic wavelets, there is another approach based on p-adic wavelet set theory introduced by J. J. Benedetto and R. L. Benedetto [3], [2], we did not see up to now orthogonal p-adic wavelet bases which are not generated by the Haar MRA.

The simplest way to construct a multivariate wavelet basis is the following. Given an univariate MRA with a scaling function φ, using the method suggested by Y. Meyer [14] (see, e.g. [15, Sec 2.1]), one can easily construct a d-dimentional separable MRA, where the scaling function is $\varphi \otimes \cdots \otimes \varphi$. If ψ is a wavelet function in the univariate MRA, then the functions $f_1 \otimes \cdots \otimes f_d$, where each f_k equals either φ or ψ and not all f_k, $k = 1, \ldots, d$, are equal to φ, form a set of wavelet functions generating a multivariate wavelet basis (i.e. the basis consists of the dilations and shifts of the wavelet functions). This approach was realized for p-adics in [17]. Next we are going to discuss only the case $p = d = 2$. The separable wavelet basis is generated by three wavalet functions in this case. E. King and one of the authors studied two-

2

dimensional Haar MRA with quincunx dilation in [6]. The basis is generated by only one wavelet function in this case. In the present paper we describe all wavelet functions corresponding to the quincunx MRA and generating an orthogonal basis. We also discuss how these bases related to the separable MRA.

2 Preliminaries and notations

2.1 p-adic numbers

The theory of p-adic numbers is presented in detail in many books (see, e.g.,[18], [8]). We restrict ourselves to a brief description.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the sets of positive integers, integers, real numbers, complex numbers, respectively. The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This p-adic norm is defined as follows: $|0|_p = 0$; if $x \neq 0$, $x = p^\gamma \frac{m}{n}$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m, n are not divisible by p, then $|x|_p = p^{-\gamma}$. The norm $|\cdot|_p$ satisfies the strong triangle inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$. The canonical form of any p-adic number $x \neq 0$ is

$$x = p^\gamma \sum_{j=0}^{\infty} x_j p^j \tag{1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j \in D_p := \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The fractional part $\{x\}_p$ of the number x equals by definition $p^\gamma \sum_{j=0}^{-\gamma-1} x_j p^j$. Thus, $\{x\}_p = 0$ if and only if $\gamma \geq 0$. We also set $\{0\}_p = 0$.

For every prime p, the additive group of $\mathbb{Q}_p$ is a locally compact abelian group which contains the compact open subgroup $\mathbb{Z}_p$. The group $\mathbb{Z}_p$ is the set $\{\alpha \in \mathbb{Q}_p \big| |\alpha|_p \leq 1\}$, the unit ball in $\mathbb{Q}_p$. Equivalently, $\mathbb{Z}_p$ is the

3

subgroup generated by 1. It is well-known ([16]) that since $\mathbb{Q}_p$ is a locally compact abelian group with a compact open subgroup, it has a Haar measure normalized so that the measure of $\mathbb{Z}_p$ is 1. For simplicity, we shall denote this measure by dx.

We also define the set $I_p = \{x \in \mathbb{Q}_p : \{x\} = x\}$. I_p is a set of coset representatives for $\mathbb{Q}_p / \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is open, I_p is discrete.

For any $d \geq 1$, $\mathbb{Q}_p^d$ is a vector space over $\mathbb{Q}_p$, i.e., $\mathbb{Q}_p^d$ consists of the vectors $x = (x_1, \ldots, x_d)$, where $x_j \in \mathbb{Q}_p$, $j = 1, \ldots, d$. The p-adic norm on $\mathbb{Q}_p^d$ is

$$|x|_p := \max_{1 \leq j \leq d} |x_j|_p.$$

The Haar measure dx on the field $\mathbb{Q}_p$ is extended to a Haar measure $d^d x = dx_1 \cdots dx_d$ on $\mathbb{Q}_p^d$ in the standard way.

Denote by $B_N = \{x \in \mathbb{Q}_p^d : |x|_p \leq p^N\}$ the ball of radius p^N with the center at the point 0, $N \in \mathbb{Z}$.

Finally, we define the set $I_p^d \subset \mathbb{Q}_p^d$ by $I_p^d := I_p \times \cdots \times I_p$.

2.2 Functions of p-adic variables

There are two main function theories over the p-adics. One deals with functions $\mathbb{Q}_p^d \to \mathbb{C}$ and the other with $\mathbb{Q}_p^d \to \mathbb{Q}_p$. We shall only deal with the former theory.

Since a Haar measure exists in $\mathbb{Q}_p^d$, the spaces $L^q(\mathbb{Q}_p^d)$ can be naturally introduced.

The functions $\chi_p(x) = e^{2\pi i \{x\}_p}$ are additive characters for the field $\mathbb{Q}_p$. For any set $S \subseteq \mathbb{Q}_p^d$, $\mathbb{1}_S$ denotes the characteristic function of S, i.e.

$$\mathbb{1}_S(x) = \begin{cases} 1 & ; \quad x \in S \\ 0 & ; \quad \text{else} \end{cases}.$$

4

Note that $\mathbb{1}_{\mathbb{Z}_p} \otimes \mathbb{1}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p^2}$.

2.3 p-Adic MRA

Definition 1 ([17])**.** A collection of closed spaces $V_j \subset L^2(\mathbb{Q}_p)$, $j \in \mathbb{Z}$, is called a *multiresolution analysis* (*MRA*) *in* $L^2(\mathbb{Q}_p)$ if the following axioms hold

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup\limits_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{Q}_p)$;

(c) $\bigcap\limits_{j \in \mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j \Longleftrightarrow f(p^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there exists a function $\varphi \in V_0$, called the *scaling function*, such that the system $\{\varphi(\cdot - a), a \in I_p\}$ is an orthonormal basis for V_0.

According to the standard scheme (see, e.g., [15, § 1.3]) for the construction of MRA-based wavelets, for each j, we define the space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \oplus W_j, \qquad j \in \mathbb{Z}, \tag{2}$$

where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f \in W_j \Longleftrightarrow f(p^{-1}\cdot) \in W_{j+1}, \quad \text{for all} \quad j \in \mathbb{Z} \tag{3}$$

and $W_j \perp W_k$, $j \neq k$. Taking into account axioms (b) and (c), we obtain

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{Q}_p) \quad \text{(orthogonal direct sum).} \tag{4}$$

If now we find functions $\psi_\nu \in W_0$, $\nu = 1, \ldots, r$, such that the system $\{\psi_\nu(x - a), a \in I_p, \nu = 1, \ldots, r\}$ is an orthonormal basis for W_0, then, due to (3) and (4), the system $\{p^{j/2}\psi_\nu(p^{-j} \cdot -a), a \in I_p, j \in \mathbb{Z}, \nu = 1, \ldots, r\}$,

5

is an orthonormal basis for $L^2(\mathbb{Q}_p)$. Such functions ψ_ν are called *wavelet functions* and the basis is a *wavelet basis.*

3   Separable and quincunx p-adic wavelets bases

3.1   Separable MRA

Let $\{V_j^{(\nu)}\}_{j \in \mathbb{Z}}$, $\nu = 1, \ldots, n$, be one-dimensional MRAs. We introduce subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{Q}_p^d)$ by

$$V_j = \bigotimes_{\nu=1}^{d} V_j^{(\nu)} = \overline{\mathrm{span}\{F = f_1 \otimes \cdots \otimes f_n, \ f_\nu \in V_j^{(\nu)}\}}. \tag{5}$$

Let $\varphi^{(\nu)}$ be a scaling function of ν-th MRA $\{V_j^{(\nu)}\}_j$. Set

$$\Phi = \varphi^{(1)} \otimes \cdots \otimes \varphi^{(n)}.$$

It is not difficult to check that the following theorem holds (see [17]).

Theorem 2. *Let* $\{V_j^{(\nu)}\}_{j \in \mathbb{Z}}$, $\nu = 1, \ldots, n$, *be MRAs in* $L^2(\mathbb{Q}_p)$. *Then the subspaces* V_j *of* $L^2(\mathbb{Q}_p^d)$ *defined by* (5) *satisfy the following properties:*

(a) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(b) $\cup_{j \in \mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{Q}_p^d)$;

(c) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j \iff f(p^{-1} \cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(e) *there exists a function* $\Phi \in V_0$ *such that the system* $\{\Phi(x - a), a \in I_p^d\}$, *forms an orthonormal basis for* V_0.

The collection of spaces V_j, $j \in \mathbb{Z}$, discussed in Theorem 2 is called a *separable multiresolution analysis* in $L^2(\mathbb{Q}_p^n)$, the function Φ from axiom (e) is said to be its scaling function.

Following the standard scheme (see, e.g., [15, § 2.1]), we define the wavelet space W_j as the orthogonal complement of V_j in V_{j+1}, i.e.,

$$W_j = V_{j+1} \ominus V_j, \quad j \in \mathbb{Z}.$$

Since

$$V_{j+1} = \bigotimes_{\nu=1}^{d} V_{j+1}^{(\nu)} = \bigotimes_{\nu=1}^{d} \left(V_j^{(\nu)} \oplus W_j^{(\nu)} \right)$$

$$= V_j \oplus \bigoplus_{e \subset \{1,\ldots,n\},\, e \neq \emptyset} \left(\bigotimes_{\nu \in e} W_j^{(\nu)} \right) \left(\bigotimes_{\mu \notin e} V_j^{(\mu)} \right).$$

So, the space W_j is a direct sum of $2^d - 1$ subspaces $W_{j,e}$, $e \subset \{1,\ldots,n\}$, $e \neq \emptyset$.

Let $\psi^{(\nu)}$ be a wavelet function, i.e. a function whose shifts (with respect to $a \in I_p$) form an orthonormal basis for $W_0^{(\nu)}$. It is clear that the shifts (with respect to $a \in I_p^d$) of the function

$$\Psi_e = \left(\bigotimes_{\nu \in e} \psi^{(\nu)} \right) \left(\bigotimes_{\mu \notin e} \varphi^{(\mu)} \right), \quad e \subset \{1,\ldots,n\}, \quad e \neq \emptyset, \qquad (6)$$

form an orthonormal basis for $W_{0,e}$. So, we have

$$L^2(\mathbb{Q}_p^d) = \bigoplus_{j \in \mathbb{Z}} W_j = \bigoplus_{j \in \mathbb{Z}} \left(\bigoplus_{e \subset \{1,\ldots,n\},\, e \neq \emptyset} W_{j,e} \right),$$

and the functions $p^{-dj/2} \Psi_e(p^j \cdot + a)$, $e \subset \{1,\ldots,n\}$, $e \neq \emptyset$, $j \in \mathbb{Z}$, $a \in I_p^d$, form an orthonormal basis for $L^2(\mathbb{Q}_p^d)$. A wavelet basis constructed in this way is called separable.

In particular, if $p = d = 2$, $\varphi^{(1)} = \varphi^{(2)} = \varphi$, (which implies $\{V_j^{(1)}\}_{j \in \mathbb{Z}} = \{V_j^{(2)}\}_{j \in \mathbb{Z}}$) and $\psi(x) = \chi_2(x/2)\mathbb{1}_{\mathbb{Z}_2}(x)$ is a wavelet function corresponding to the univariate MRA, then there are three wavelet functions $\Psi^1 = \varphi \otimes \psi$, $\Psi^2 = \psi \otimes \varphi$, $\Psi^3 = \psi \otimes \psi$. corresponding to the separable MRA.

3.2   Quincunx MRA

Set

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

A is the inverse of the well-known quincunx matrix. In [7] (see also [15, § 2.8]), a Haar multiresolution analysis was presented for $L^2(\mathbb{R}^2)$ which used dilations by the quincunx. Since $|\det A^{-1}| = 2$, there was only one wavelet generator. However, the support of the scaling function was a fractal, the twin dragon fractal. In contrast, it was proved in [6] that the Haar MRA for $L^2(\mathbb{Q}_2^2)$ associated to this matrix corresponds to a scaling function with a simple support, namely, $\mathbb{Z}_2^2$. To state this in more detail set

$$V_j^Q = \overline{\text{span}\{\mathbb{1}_{\mathbb{Z}_2^2}(A^j \cdot -a) : a \in I_2^2\}}. \tag{7}$$

Theorem 3 ([6]). *The subspaces V_j^Q of $L^2(\mathbb{Q}_2^2)$, $j \in \mathbb{Z}^d$, defined by (7) satisfy the following properties:*

(a) $V_j^Q \subset V_{j+1}^Q$ for all $j \in \mathbb{Z}$,

(b) $\bigcup_{j \in \mathbb{Z}} V_j^Q$ is dense in $L^2(\mathbb{Q}_2^2)$,

c() $\bigcap_{j \in \mathbb{Z}} V_j^Q = \{0\}$,

(d) $f \in V_j^Q \Leftrightarrow f(A\cdot) \in V_{j+1}^Q$ for all $j \in \mathbb{Z}$,

(e) there exists a function $\phi \in V_0^Q$, such that the system $\{\phi(\cdot - a) : a \in I_2^2\}$ is an orthonormal basis for V_0^Q.

The collection of spaces V_j^Q, $j \in \mathbb{Z}$, discussed in Theorem 3 is called the *quincunx Haar multiresolution analysis* in $L^2(\mathbb{Q}_p^n)$, the function ϕ from axiom (e) is its scaling function. It follows from (7) that $\phi = \mathbb{1}_{\mathbb{Z}_2^2}$.

It was proved in [6] that the function

$$\psi = \phi(A\cdot) - \phi\left(A\cdot - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) \tag{8}$$

is a wavelet function corresponding to the quincunx MRA, i.e. the system $\{\psi(\cdot - a), a \in I_p^2\}$ is an orthonormal basis for the wavelet space $W_0^Q := V_1^Q \ominus V_0^Q$. It follows that the system

$$\{\psi(A^j \cdot -a), \quad j \in \mathbb{Z}, a \in I_2^2\} \tag{9}$$

is an orthonormal basis for $L^2(\mathbb{Q}_2^2)$.

Next we shall show that this basis corresponds also to the separable MRA in some sense. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a separable MRA with $p = d = 2$, $\varphi^{(1)} = \varphi^{(2)} = \mathbb{1}_{\mathbb{Z}_2}$, i.e. both $\{V_j^{(1)}\}_{j\in\mathbb{Z}}$ and $\{V_j^{(2)}\}_{j\in\mathbb{Z}}$ are Haar MRAs. In this case, $\Phi = \mathbb{1}_{\mathbb{Z}_2} \otimes \mathbb{1}_{\mathbb{Z}_2} = \mathbb{1}_{\mathbb{Z}_2^2}$. Thus, $\Phi = \phi$, which yields $V_0 = V_0^Q$. Since

$$A^2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

we have

$$\phi(A^2 x - a) = \phi\left(\begin{pmatrix} \frac{x_2}{2} - a_1 \\ \frac{x_1}{2} - a_2 \end{pmatrix}\right) = \phi\left(\begin{pmatrix} \frac{x_1}{2} - a_2 \\ \frac{x_2}{2} - a_1 \end{pmatrix}\right).$$

It follows that

$$\{\phi(A^2 x - a), \quad a \in I_2^2\} = \{\phi(\frac{x}{2} - a), \quad a \in I_2^2\}.$$

So, $V_0^Q \oplus W_0^Q \oplus W_1^Q = V_2^Q = V_1$, which yields that $W_0 = W_0^Q \oplus W_1^Q$, the functions $\psi(\cdot - a), \psi(A \cdot -a), \quad a \in I_2^2$, form an orthonormal basis for W_0 and the dyadic dilations of these functions form an orthonormal basis for $L^2(\mathbb{Q}_2^2)$. These relations may be treated as follows: the quincunx Haar MRA is the "square root" of the separable Haar MRA.

9

4   Description of wavelet functions

As was said above, (9) is an orthonormal basis for $L^2(\mathbb{Q}_2^2)$ generated by a single wavelet function ψ. In contrast to the real setting, p-adic wavelet function corresponding to an MRA is not unique. Now we are going to find other wavelet functions corresponding to the quincunx Haar MRA.

First we shall prove some auxiliary facts.

Lemma 4. Let ψ, $\tilde{\psi}$ be compactly supported functions with their supports contained in B_s, $s \in \mathbb{N}$, and such that

$$\tilde{\psi}(x) = \sum_{a \in I_2^2} h_a \psi(x - a). \tag{10}$$

Then $h_a = 0$ whenever $|a|_2 > 2^s$.

Proof. We rewrite (10) in the form

$$\tilde{\psi}(x) = \sum_{\substack{a \in I_2^2 \\ |a|_2 \leq 2^s}} h_a \psi(x - a) + \sum_{\substack{a \in I_2^2 \\ |a|_2 > 2^s}} h_a \psi(x - a).$$

The second sum on the right-hand side vanishes whenever $|x|_2 \leq 2^s$ because $\operatorname{supp} \psi \subset B_s$. The first sum on the right-hand side and the left-hand side vanish whenever $|x|_2 > 2^s$ because $\operatorname{supp} \psi \subset B_s$ and $\operatorname{supp} \tilde{\psi} \subset B_s$ respectively. So, the second sum is equal to zero for all $x \in \mathbb{Q}_2^2$. $\Diamond$

Lemma 5. Let $n \in \mathbb{Z}_2^2$, and let ψ be the function defined in (8). If $An \in \mathbb{Z}_2^2$, then $\psi(\cdot + n) = \psi$, if $An \notin \mathbb{Z}_2^2$, then $\psi(\cdot + n) = -\psi$.

Proof. Observe that $An \in \mathbb{Z}_2^2$ if and only if both coordinates of n are either even or odd. Therefore, if $An \in \mathbb{Z}_2^2$, then both coordinates of An are integers, otherwise both coordinates of An are semi-integers. It remains to note that the function ϕ is $1 - periodic$ with respect to each variable. $\Diamond$

Theorem 6. *All compactly supported wavelet functions corresponding to the quincunx Haar MRA are given by*

$$\tilde{\psi}(x) = \sum_{k=0}^{2^s-1} \sum_{l=0}^{2^s-1} \alpha_{kl} \psi\left(x - \binom{k/2^s}{l/2^s}\right) \tag{11}$$

where $s \in \mathbb{N}$, ψ is the function defined by (8),

$$\alpha_{kl} = \begin{cases} 2^{-2s}(-1)^k e^{-\frac{\pi i k}{2^s}} \displaystyle\sum_{p=0}^{2^s-1}\sum_{q=0}^{2^s-1} e^{-2\pi i \frac{qk}{2^s}} \gamma_{qp}, & if \quad l = 0, \\[4mm] 2^{-2s}(-1)^{k-l+1} e^{-\frac{\pi i (k-l)}{2^s}} \displaystyle\sum_{p=0}^{2^s-1}\sum_{q=0}^{2^s-1} e^{-2\pi i \frac{qk-lp}{2^s}} \gamma_{qp}, & if \quad l \neq 0, \end{cases} \tag{12}$$

$$\gamma_{qp} \in \mathbb{C}, \quad |\gamma_{qp}| = 1, \quad p, q = 0, \ldots, 2^s - 1.$$

Proof. Let $\tilde{\psi}$ be a compactly supported wavelet function corresponding to the quincunx Haar MRA, i.e., the system $\{\tilde{\psi}(\cdot - a), \quad a \in I_2^2\}$ is an orthonormal basis for W_0^Q, and supp $\tilde{\psi} \subset B_s$. Since $\{\psi(\cdot - a), \quad a \in I_2^2\}$ is a basis for W_0^Q and $\tilde{\psi} \in W_0^Q$, we have

$$\tilde{\psi}(x) = \sum_{a \in I_2^2} h_a \psi(x - a).$$

Due to Lemma 4, taking into account that supp $\psi \subset B_1 \subset B_s$, supp $\tilde{\psi} \subset B_s$, we obtain (11) with some coefficients α_{kl}.

Since $\{\psi(\cdot - a), \ a \in I_2^2\}$ is an orthogonal system, evidently, $\tilde{\psi}$ is orthogonal to $\tilde{\psi}(\cdot - a)$ whenever $a \in I_2^2$ and $a \neq \binom{k/2^s}{l/2^s}, k, l = 0, 1, \ldots, 2^s - 1$. Thus, the system $\{\tilde{\psi}(\cdot - a), \quad a \in I_2^2\}$ is an orthonormal system if and only if the system consisting of the functions $\tilde{\psi}\left(x - \binom{k/2^s}{l/2^s}\right)$, $k, l = 0, 1, \ldots, 2^s - 1$, is orthonormal.

11

Next we need the following notations. Set

$$
\Lambda = \begin{pmatrix} \mathbb{A} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{A} & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{A} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & -\mathbb{I} \\ \mathbb{I} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{I} & \mathbb{O} \end{pmatrix},
$$

where $\mathbb{O}, \mathbb{I}, \mathbb{A}$ are $2^s \times 2^s$ matrices, $\mathbb{O}$ is the zero matrix, $\mathbb{I}$ is the unity matrix,

$$
\mathbb{A} = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.
$$

Let α be a 2^{2s}-dimensional vector (column) whose N-th coordinate, $N = 2^s l + k$, $k, l = 0, 1 \ldots 2^s - 1$, is α_{kl}, and let Ψ, $\tilde{\Psi}$ be 2^{2s}-dimensional vector-functions (columns) whose N-th coordinates, $N = 2^s l + k$, $k, l = 0, 1, \ldots, 2^s - 1$, are $\psi\left(x - \binom{k/2^s}{l/2^s}\right)$ and $\tilde{\psi}\left(x - \binom{k/2^s}{l/2^s}\right)$ respectively. Using Lemma 5, we obtain $\tilde{\Psi} = D\Psi$, where D is a $2^{2s} \times 2^{2s}$ matrix whose N-th row, $N = 2^s l + k$, $k, l = 0, 1, \ldots, 2^s - 1$, is $\Omega^l \Lambda^k a$.

Due to orthonormality of the system $\{\psi(\cdot - a), \ a \in I_2^2\}$, the coordinates of $\tilde{\Psi}$ form an orthonormal system if and only if the matrix D is unitary. To describe all unitary matrices D we need to find all $\alpha \in \mathbb{C}^{2^{2s}}$ such that the vectors $\alpha, \Lambda\alpha, \ldots, \Omega^l \Lambda^k \alpha, \ldots, \Omega^{2^s - 1} \Lambda^{2^s - 1} \alpha$ are orthonormal. We have already one such vector $\alpha_0 = (1, 0, \ldots, 0)^T$ because the matrix D_0 whose rows are $\alpha_0, \Lambda\alpha_0, \ldots, \Omega^l \Lambda^k \alpha_0, \ldots, \Omega^{2^s - 1} \Lambda^{2^s - 1} \alpha_0$ is, evidently, unitary.

Let us prove that the rows of D are orthonormal if and only if $\alpha = B\alpha_0$,

where B is a unitary matrix such that

$$\Lambda B = B\Lambda, \tag{13}$$

$$\Omega B = B\Omega. \tag{14}$$

Indeed, let $\alpha = B\alpha_0$, with B being a unitary matrix satisfying (13), (14). In this case $\Omega^l \Lambda^k \alpha = \Omega^l \Lambda^k B\alpha_0 = B(\Omega^l \Lambda^k \alpha_0)$, and the unitarity of D_0 implies the unitarity of D. Conversly, if the rows of D are orthonormal, taking into account that the rows of D_0 are also orthonormal, we conclude that there exists a unitary matrix B such that

$$D^T = BD_0^T \tag{15}$$

So we have, in particular,

$$\alpha = B\alpha_0, \quad \Lambda\alpha = B\Lambda\alpha_0, \quad \Lambda^2\alpha = B\Lambda^2\alpha_0, \dots, \Lambda^{2^s-1}\alpha = B\Lambda^{2^s-1}\alpha_0.$$

Substituting the first equality into the second one, we obtain $(\Lambda B - B\Lambda)\alpha_0 = 0$. Using this and substituting the first equality into the third one, we obtain $(\Lambda B - B\Lambda)(\Lambda\alpha_0) = 0$. After similar manipulations with other equalities we have

$$(\Lambda B - B\Lambda)(\Lambda^p \alpha_0) = 0, \quad p = 0, 1, \dots, 2^s - 2. \tag{16}$$

Since $\Lambda^{2^s}\alpha = -\alpha$, $\Lambda^{2^s}\alpha_0 = -\alpha_0$, substituting $\alpha = B\alpha_0$ and using (16) we obtain $(\Lambda B - B\Lambda)(\Lambda^{2^s-1}\alpha_0) = 0$. Next it follows from (15) that

$$\Omega\alpha = B\alpha_0, \quad \Omega\Lambda\alpha = B\Omega\Lambda\alpha_0, \quad \Omega\Lambda^2\alpha = B\Omega\Lambda^2\alpha_0, \dots, \Omega\Lambda^{2^s-1}\alpha = B\Omega\Lambda^{2^s-1}\alpha_0.$$

Substituting $\alpha = B\alpha_0$ into the first equality, we have $(\Omega B - B\Omega)\alpha_0 = 0$. Using this, taking into account that $\Omega\Lambda = \Lambda\Omega$ and substituting $\alpha = B\alpha_0$ into the second equality, we obtain that $(\Lambda B - B\Lambda)(\Omega\alpha_0) = 0$. After similar

manipulations with the other equalities and with the equalities $\Omega\Lambda^{2^s}\alpha = -\Omega\alpha$, $\Omega\Lambda^{2^s}\alpha_0 = -\Omega\alpha_0$, we have

$$(\Lambda B - B\Lambda)(\Lambda^p \Omega^q \alpha_0) = 0, \quad p = 0, 1, \ldots, 2^s - 1.$$

Continuing this process, we get that $(\Lambda B - B\Lambda)(\Lambda^p \alpha_0) = 0$ for all $p, q = 0, 1, \ldots, 2^s - 1$. Since $\{\Lambda^p \Omega^q \alpha_0\}_{p,q=0}^{2^s-1}$ is a basis for $\mathbb{C}^{2^{2s}}$, we obtain (13). In a similar way we can check that (14) holds.

Thus we should describe all unitary matrices B satisfying (13), (14). Let

$$B = \begin{pmatrix} B_{00} & \cdots & B_{0,2^s-1} \\ \vdots & \ddots & \vdots \\ B_{2^s-1,0} & \cdots & B_{2^s-1,2^s-1} \end{pmatrix},$$

where B_{ij} is a $2^s \times 2^s$ matrix. It follows from (14) that

$$\begin{pmatrix} -B_{2^s-1,0} & -B_{2^s-1,1} & \cdots & -B_{2^s-1,2^s-1} \\ B_{00} & B_{01} & \cdots & B_{0,2^s-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{2^s-2,0} & B_{2^s-2,1} & \cdots & B_{2^s-2,2^s-1} \end{pmatrix} = \begin{pmatrix} B_{01} & \cdots & B_{0,2^s-1} & -B_{00} \\ B_{11} & \cdots & B_{1,2^s-1} & -B_{10} \\ \vdots & \ddots & \vdots & \vdots \\ B_{2^s-1,1} & \cdots & B_{2^s-1,2^s-1} & -B_{2^s-1,0} \end{pmatrix}.$$

This yields

$$B_{00} = B_{11} = \cdots = B_{2^s-1,2^s-1},$$

$$B_{0l} = B_{1,l+1} = \cdots = B_{2^s-l-1,2^s-1} =$$

$$-B_{2^s-l,0} = \cdots = -B_{2^s-1,l-1}, \quad l = 1, \ldots, 2^s - 1.$$

i.e.

$$B = \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_{2^s-1} \\ -\beta_{2^s-1} & \beta_0 & \cdots & \beta_{2^s-2} \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_1 & -\beta_2 & \cdots & \beta_0 \end{pmatrix},$$

14

where $\beta_l = B_{0l}$. It is not difficult to see that any such matrix B satisfies (14).

Since

$$\Lambda B = \begin{pmatrix} \mathbb{A}\beta_0 & \mathbb{A}\beta_1 & \dots & \mathbb{A}\beta_{2^s-1} \\ \mathbb{A}(-\beta_{2^s-1}) & \mathbb{A}\beta_0 & \dots & \mathbb{A}\beta_{2^s-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{A}(-\beta_1) & \mathbb{A}(-\beta_2) & \dots & \mathbb{A}\beta_0 \end{pmatrix}, \quad B\Lambda = \begin{pmatrix} \beta_0\mathbb{A} & \beta_1\mathbb{A} & \dots & \beta_{2^s-1}\mathbb{A} \\ (-\beta_{2^s-1})\mathbb{A} & \beta_0\mathbb{A} & \dots & \beta_{2^s-2}\mathbb{A} \\ \vdots & \vdots & \ddots & \vdots \\ (-\beta_1)\mathbb{A} & (-\beta_2)\mathbb{A} & \dots & \beta_0\mathbb{A}, \end{pmatrix}$$

(13) holds if and only if $\beta_\nu \mathbb{A} = \mathbb{A}\beta_\nu$, for all $\nu = 0, \dots, 2^s - 1$. It follows from the proof of Theorem 4.1 in [17] that β_n commutes with $\mathbb{A}$ if and only if $\beta_\nu = C\tilde{\beta}_\nu C^{-1}$, where $\tilde{\beta}_\nu$ is a diagonal matrix, C is a matrix whose entries are given by

$$c_{pq} = 2^{-s/2}(-1)^p e^{-\pi i p \frac{2q+1}{2^s}}, \quad p,q = 0, \dots, 2^s - 1. \tag{17}$$

Since C is a unitary matrix, B is unitary if and only if the matrix

$$\tilde{B} = \begin{pmatrix} \tilde{\beta}_0 & \tilde{\beta}_1 & \dots & \tilde{\beta}_{2^s-1} \\ -\tilde{\beta}_{2^s-1} & \tilde{\beta}_0 & \dots & \tilde{\beta}_{2^s-2} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_1 & -\tilde{\beta}_2 & \dots & \tilde{\beta}_0 \end{pmatrix}$$

is unitary. Set $\theta_l = (\lambda_l^{(0)}, \dots, \lambda_l^{(2^s-1)})^T$, where $\lambda_l^{(\nu)}$ is the l-th diagonal element of $\tilde{\beta}_\nu$. It is easy to see that $\tilde{B}$ is unitary if and only if the vectors $\theta_l, \mathbb{A}\theta_l, \dots, \mathbb{A}^{2^s-1}\theta_l$ form an orthonormal system for all $l = 0, \dots, 2^s - 1$. It follows from the proof of Theorem 4.1 in [17] that all such θ_l are given by

$$\theta_l = C \begin{pmatrix} \gamma_{l0} & 0 & \dots & 0 \\ 0 & \gamma_{l1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{l,2^s-1} \end{pmatrix} C^{-1} e_1,$$

where $\gamma_{lk} \in \mathbb{C}$, $|\gamma_{lk}| = 1$, $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{2^s}$. Substituting (17), we have

$$\lambda_l^{(\nu)} = 2^{-s}(-1)^\nu e^{-\pi i \frac{\nu}{2^s}} \sum_{k=0}^{2^s-1} e^{-2\pi i \frac{k\nu}{2^s}} \gamma_{lk}, \quad l, \nu = 0, \ldots, 2^s - 1. \qquad (18)$$

Again using (17), we obtain

$$\alpha = B\alpha_0 = \begin{pmatrix} C & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & C & \ldots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & C \end{pmatrix} \tilde{B} \begin{pmatrix} C^{-1} & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & C^{-1} & \ldots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & C^{-1} \end{pmatrix} \alpha_0 =$$

$$\begin{pmatrix} C & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & C & \ldots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & C \end{pmatrix} \begin{pmatrix} \delta_0 \\ -\delta_{2^s-1} \\ \vdots \\ -\delta_1 \end{pmatrix}, \qquad (19)$$

where

$$\delta_\nu = \tilde{\beta}_\nu \begin{pmatrix} \overline{c_{01}} \\ \vdots \\ \overline{c_{0,2^s-1}} \end{pmatrix} = 2^{-s/2} \tilde{\beta}_\nu \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 2^{-s/2} \begin{pmatrix} \lambda_0^{(\nu)} \\ \vdots \\ \lambda_{2^s-1}^{(\nu)} \end{pmatrix}. \qquad (20)$$

To prove (12) it remains to combine (19) with (20), (18), and take into account that α_{0k} is the k-th coordinate of $C\delta_0$ and α_{lk} is the k-th coordinate of $C\delta_{2^s-l}$ for $l = 1, \ldots, 2^s - 1$. $\Diamond$

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